Congruence Closure with Free Variables

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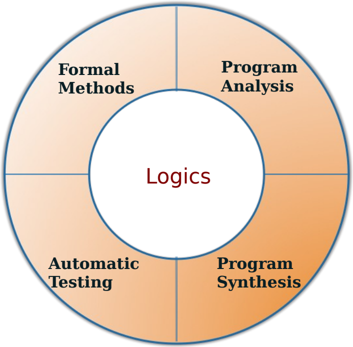
SMT solvers are successfully used in a variety of applications, including many verification tools



Picture credit: Vijay Ganesh

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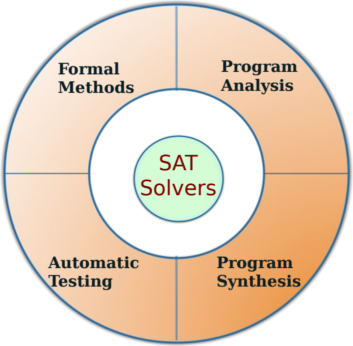

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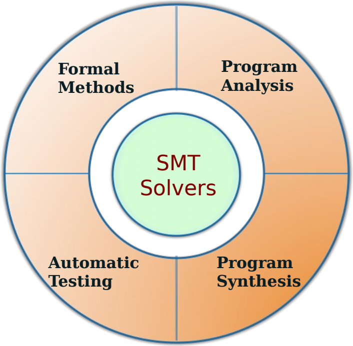

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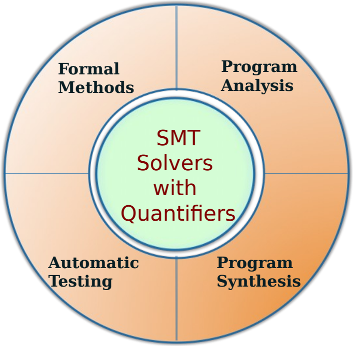

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Quantifiers in SMT solvers

Quantifiers primarily handled with heuristic instantiation

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Quantifiers in SMT solvers

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Fast semantically guided instantiation techniques

- ⊖ ~~Too many instances swamp solver~~ **Fewer, necessary instances**
- ⊖ ~~Butterfly effect~~ **Reduce dependency on heuristics**

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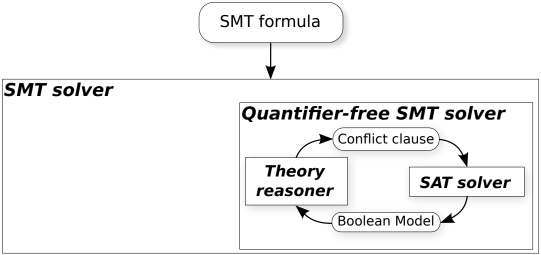
- ~~A ground model with 10^2 ground each applications for f, g, h leads to up to 10^6 instantiations~~
- **Derive instantiations that refute ground model**

Problem statement



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  - E is a conjunctive set of ground literals
  - $\mathcal{Q}$ is a conjunctive set of quantified clauses

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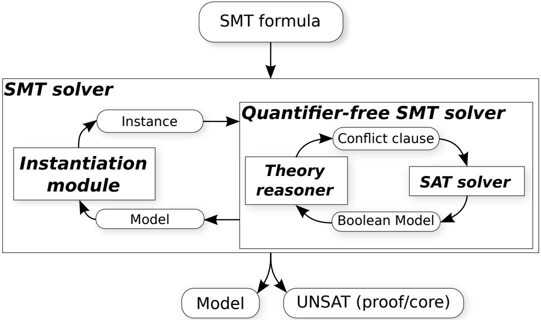

- ▷ Quantifier-free solver enumerates models $E \cup \mathcal{Q}$
  - ▶ E is a conjunctive set of ground literals
  - ▶ $\mathcal{Q}$ is a conjunctive set of quantified clauses
- ▷ Instantiation module generates instances from $\mathcal{Q}$ and adds them to E

Heuristic instantiation

Pattern-matching of terms from $\mathcal{Q}$ into terms of E

No consistency check of $E \cup \mathcal{Q}$

- ⊖ Fast, but too many instances

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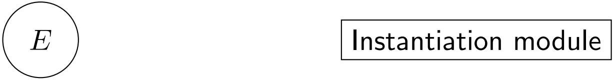

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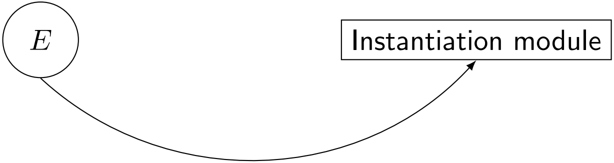

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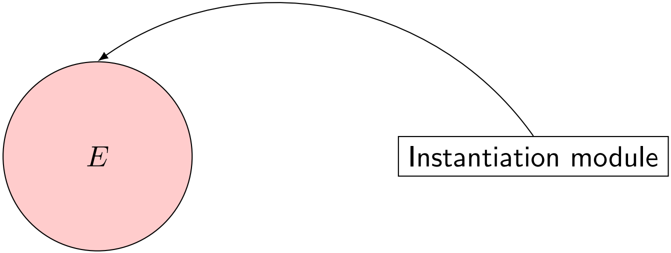

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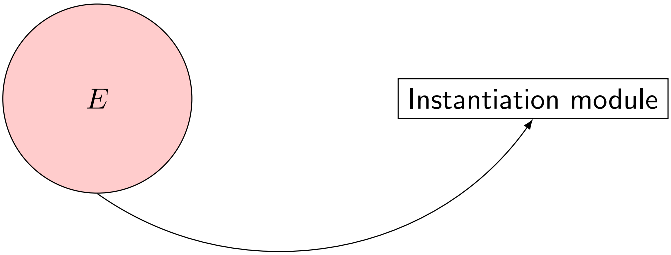

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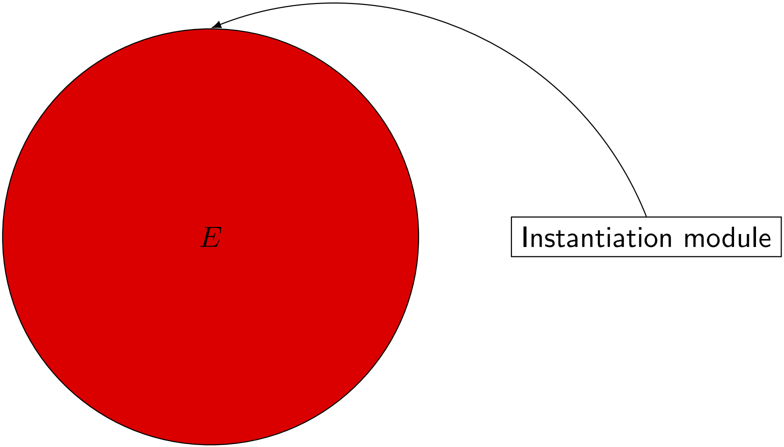

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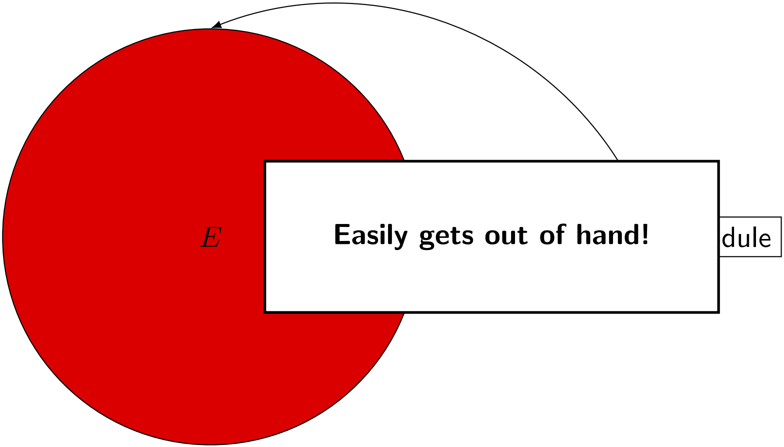

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Check consistency of $E \cup \mathcal{Q}$

- $\oplus$ Only instances refuting the current model are generated

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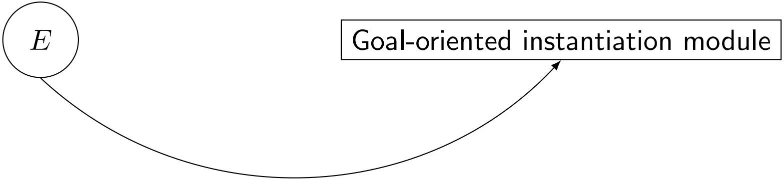

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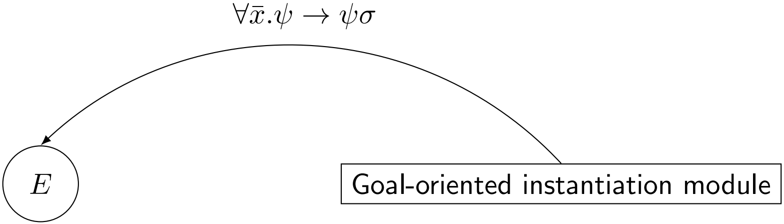

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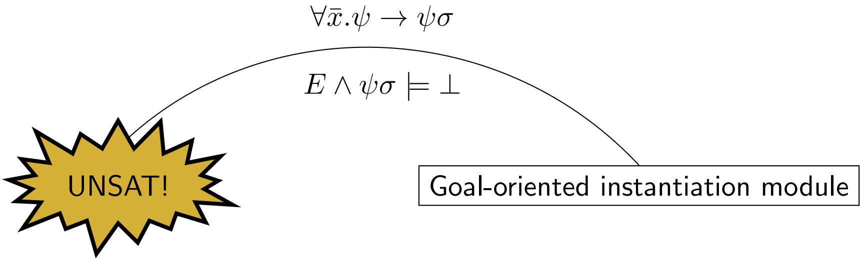

Previous work

Conflict-based instantiation [RTM14]

- ▷ Given a model $E \cup \mathcal{Q}$, for some $\forall \bar{x}.\ \psi \in \mathcal{Q}$ find σ s.t. $E \wedge \psi\sigma \models \bot$
- ▷ Add instance $\forall \bar{x}.\ \psi \rightarrow \psi\sigma$ to quantifier-free solver

Finding conflicting instances requires deriving σ s.t.

$$E \models \neg\psi\sigma$$

- ⊕ Goal-oriented
- ⊕ Efficient
- ⊖ Ad-hoc
- ⊖ Incomplete

Let's look deeper into the problem

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Given conjunctive sets of equality literals E and L, with E ground, finding a substitution σ s.t. $E \models L\sigma$

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- ▷ Variant of classic (non-simultaneous) rigid E-unification

- ▷ NP-complete

  NP: Solutions can be restricted to ground terms in $E \cup L$
  NP-hard: reduction of 3-SAT

Congruence Closure with Free Variables (CCFV)

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CCFV is a sound, complete and terminating calculus for solving E-ground (dis)unification

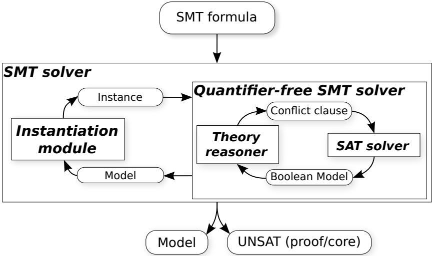
- ⊕ Goal-oriented
- ⊕ **(More)** Efficient
- ⊖ ~~Ad-hoc~~ **Versatile framework, recasting many instantiation techniques as a CCFV problem**
- ⊖ ~~Incomplete~~ **Finds all conflicting instances of a quantified formula**

Existing techniques as special cases

- ▷ Conflict-based instantiation [RTM14]
  - ⊕ CCFV provides formal guarantees and more clear extensions
- ▷ E-matching based heuristic instantiation [DNS05; MB07]
  - ⊕ CCFV allows to easily discard instances already entailed by E
- ▷ Model-based instantiation [GM09; RTG+13]
  - ⊕ No need for a secondary ground SMT solver
  - ⊕ No need to guess solutions

Towards a theory solver for instantiation

- Model generation
- **Conflict set generation**
- **Propagation**
- Incrementality



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Finding solutions σ for $E \models L\sigma$

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- Different possibilities for building solutions are handled with branching and backtracking

$$
\begin{array}{rcl}
E & \models & L\sigma \\
f(a) \simeq f(b) \land g(b) \not\simeq h(c) & \models & (f(x) \simeq f(z) \land h(y) \not\simeq g(z))\,\sigma
\end{array}
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$$f(x) \simeq f(z) \wedge h(y) \not\simeq g(z)$$

$$E \models L\sigma$$

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$$\varnothing \;\Big|$$

$$y \simeq c \wedge z \simeq b \wedge f(x) \simeq f(z)$$

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$$y \simeq c$$

$$z \simeq b \wedge f(x) \simeq f(z)$$

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$$f(x) \simeq f(z) \wedge h(y) \not\simeq g(z)$$

$$\emptyset$$

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$$y \simeq c$$

$$z \simeq b \wedge f(x) \simeq f(z)$$

$$y \simeq c,\; z \simeq b$$

$$f(x) \simeq f(b)$$

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$x \simeq a$ | $x \simeq b$

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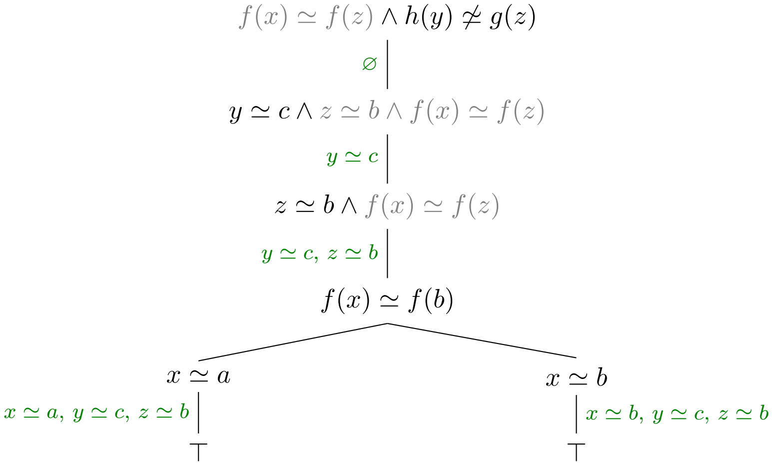
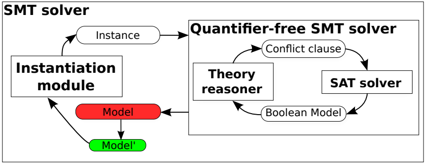


Implementation

- Model minimisation



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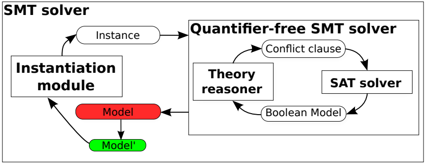

- ▷ Top symbol indexing of E-graph from ground congruence closure

$$E \models f(x)\sigma \simeq t \text{ only if } [t] \text{ contains some } f(t')$$

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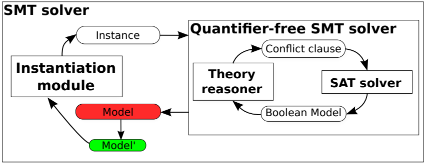

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$$f \to \begin{cases} f([t_1], \ldots, [t_n]) \\ \ldots \\ f([t'_1], \ldots, [t'_n]) \end{cases}$$

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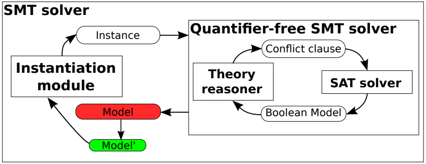

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$$f \rightarrow \begin{cases} f([t_1], \ldots, [t_n]) \\ \ldots \\ f([t'_1], \ldots, [t'_n]) \end{cases}$$

- Bitsets for fast checking if a symbol has applications in a congruence class

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- Selection strategies

$$E \models f(x, y) \simeq h(z) \wedge x \simeq t \wedge C$$

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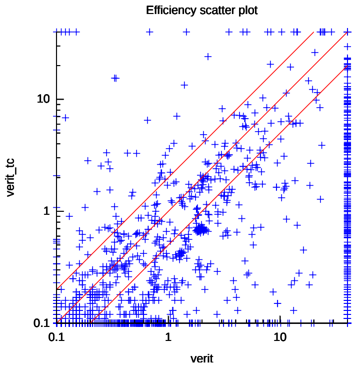
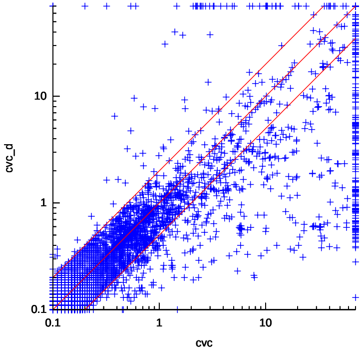
$$E \models f(t, y) \simeq h(z) \wedge C$$

  - ▶ $E \models f(t, y)\sigma \simeq h(z)\sigma$ only if there is some $f(t', t'')$ s.t.

$$E \models t \simeq t'$$





veriT: + 800 out of $1\,785$ unsolved problems

CVC4: + 200 out of 745 unsolved problems

* experiments in the "UF", "UFLIA", "UFLRA" and "UFIDL" categories of SMT-LIB, which have $10\,495$ benchmarks annotated as unsatisfiable, with 30s timeout.

Conclusions and future work

- ▷ A unifying framework for quantified formulas with equality and uninterpreted functions
- ▷ Lifting congruence closure to accommodate free variables
- ▷ Efficient implementations in the SMT solvers CVC4 and veriT

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- A unifying framework for quantified formulas with equality and uninterpreted functions
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Extensions

- Finding conflicting instances across multiple quantified formulas

$$E \models \neg\psi_1\sigma \vee \cdots \vee \neg\psi_n\sigma, \quad \forall \bar{x}.\ \psi \in \mathcal{Q}$$

- Incrementality
- Learning-based search for solutions
- Beyond theory of equality
- Handle variables in E

Thank you

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