Scalable fine-grained proofs for formula processing

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Why proofs?

- check results for unsatisfiable/valid formulas
- export proofs into other tools

- debugging
- evaluate algorithms

- take part to the reasoning framework (e.g. conflict clauses)
- extract cores
- compute interpolants

But what kind of proofs?

Challenges for proofs in automated reasoning

- Collecting and storing proof information efficiently
  still an issue, lots of work in progress
  [KBT+16; HBR+15; KV13; Sch13; BODF09; WDF+09; Mos08; MB08; SZS04]
- Producing proofs for sophisticated processing techniques
  proofs with holes or too coarse
- Extract proofs for some decision procedures (e.g. CAD)
  still a research subject
- Standardizing a proof format
  open

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  ~~proofs with holes or too coarse~~ **scalable fine-grained proofs**

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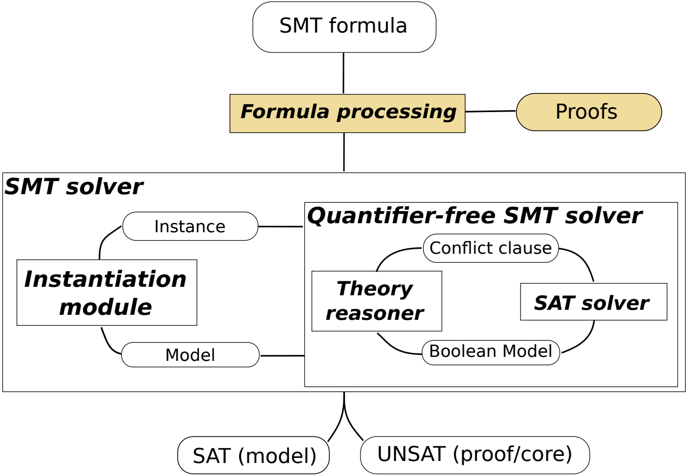
SMT: the big picture



Proofs in SMT: main reasoning steps

Resolution chains, input formulas, tautologies for theory and quantifier reasoning

- SAT solver: resolution

$$\frac{A \vee \ell \qquad B \vee \bar{\ell}}{A \vee B}$$

Antecedents: $A \vee \ell$, $B \vee \bar{\ell}$
Pivot: ℓ or $\bar{\ell}$
Resolvent: $A \vee B = (A \vee \ell) \diamond (B \vee \bar{\ell})$

- theory solvers: theory lemmas

$$\neg(a \simeq c) \vee \neg(c \simeq b) \vee a \simeq b \qquad \neg(a \simeq b) \vee f(a) \simeq f(b)$$

$$\neg(y > 1) \vee \neg(x < 1) \vee y > x$$

- instantiation module: instantiation lemmas

$$\neg(\forall x.\, \psi[x]) \vee \psi[t]$$

Proving formula processing

- ⊖ Resolution does not capture all transformations
- ⊖ Some transformations do not preserve logical equivalence
- ⊖ Processing code is lengthy and deals with many cases
- ⊖ Difficult to manipulate binders soundly and efficiently

Proving formula processing

- ~~Resolution does not capture all transformations~~
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- ~~Difficult to manipulate binders soundly and efficiently~~

Extensible framework to produce proofs for processing techniques involving *locally replacing equals by equals* in the presence of *binders*

Some instances:

Skolemization: $(\neg\,\forall x.\ \mathsf{p}(x)) \simeq \neg\,\mathsf{p}(\varepsilon x.\ \neg\,\mathsf{p}(x))$

let elimination: $(\mathrm{let}\ x \simeq \mathsf{a}\ \mathrm{in}\ \mathsf{p}(x,\ x)) \simeq \mathsf{p}(\mathsf{a},\ \mathsf{a})$

theory simplifications: $(\mathsf{k} + 1 \times 0 < \mathsf{k}) \simeq (\mathsf{k} < \mathsf{k})$

Inference system

A context Γ fixes a set of variables and specifies a substitution

substitution

$$\Gamma ::= \varnothing \mid \Gamma, x \mid \Gamma, \bar{x}_n \mapsto \bar{s}_n$$

bound variable

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substitution

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Rules have the form

derivations of premises

$$\frac{\mathcal{D}_1 \quad \cdots \quad \mathcal{D}_n}{\Gamma \rhd t \simeq u}\ \mathrm{R}$$

assumptions

transformation

- ▷ Semantically, the judgement expresses the equality of the terms $\Gamma(t)$ and u for all variables fixed by Γ

Inserting processing proofs into resolution proof

Now assertions may have different justifications other than tautologies and resolution:

$$\dfrac{\varphi \qquad \dfrac{\mathcal{D}}{\varphi \simeq \psi} \qquad \dfrac{}{\neg\,(\varphi \simeq \psi) \vee \neg\,\varphi \vee \psi}\ \textsc{Taut}_{\simeq}}{\psi}\ \textsc{Resolve}$$

in which $\mathcal{D}$ is the derivation of the processing of φ into ψ, which yields the conclusion $\varphi \simeq \psi$

Example of theory simplification

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\rhd\ \mathsf{k} \simeq \mathsf{k}}\ \textsc{Cong} \qquad \dfrac{}{\rhd\ 1\times 0 \simeq 0}\ \textsc{Taut}_\times}{\rhd\ \mathsf{k}+1\times 0 \simeq \mathsf{k}+0}\ \textsc{Cong} \qquad \dfrac{}{\rhd\ \mathsf{k}+0 \simeq \mathsf{k}}\ \textsc{Taut}_+}{\rhd\ \mathsf{k}+1\times 0 \simeq \mathsf{k}}\ \textsc{Trans} \qquad \dfrac{}{\rhd\ \mathsf{k} \simeq \mathsf{k}}\ \textsc{Cong}}{\rhd\ (\mathsf{k}+1\times 0 < \mathsf{k}) \simeq (\mathsf{k} < \mathsf{k})}\ \textsc{Cong}$$

Output skolemization

The skolemization proof of the formula $\neg \forall x.\, \mathsf{p}(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \mapsto \varepsilon x.\, \neg \mathsf{p}(x) \;\rhd\; x \simeq \varepsilon x.\, \neg \mathsf{p}(x)}\ \textsc{Refl}}{x \mapsto \varepsilon x.\, \neg \mathsf{p}(x) \;\rhd\; \mathsf{p}(x) \simeq \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x))}\ \textsc{Cong}}{\rhd\; (\forall x.\, \mathsf{p}(x)) \simeq \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x))}\ \textsc{Sko}_\forall}{\rhd\; (\neg \forall x.\, \mathsf{p}(x)) \simeq \neg \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x))}\ \textsc{Cong}$$

veriT syntax:

$$\begin{array}{l}
(.\mathrm{c0}\ (\mathsf{Sko_All}\ \ {:}\mathrm{conclusion}\ ((\forall x.\, \mathsf{p}(x)) \simeq \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x)))\\
\qquad\quad {:}\mathrm{args}\ (x \mapsto (\varepsilon x.\, \neg \mathsf{p}(x)))\\
\qquad\quad {:}\mathrm{subproof}\ ((.\mathrm{c1}\ (\mathsf{Refl}\ \ {:}\mathrm{conclusion}\ (x \simeq (\varepsilon x.\, \neg \mathsf{p}(x)))))\\
\qquad\qquad\qquad\quad (.\mathrm{c2}\ (\mathsf{Cong}\ \ {:}\mathrm{clauses}\ (.\mathrm{c1})\\
\qquad\qquad\qquad\qquad\qquad\quad {:}\mathrm{conclusion}\ (\mathsf{p}(x) \simeq \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x))))))))\\
(.\mathrm{c3}\ (\mathsf{Cong}\ \ {:}\mathrm{clauses}\ (.\mathrm{c0})\ \ {:}\mathrm{conclusion}\ ((\neg \forall x.\, \mathsf{p}(x)) \simeq \neg \mathsf{p}(\varepsilon x.\, \neg \mathsf{p}(x)))))
\end{array}$$

Example of ‘let’ expansion

$$
\dfrac{\dfrac{}{\rhd\ \mathsf{a} \simeq \mathsf{a}}\ \textsc{Cong} \qquad \dfrac{\dfrac{}{x \mapsto \mathsf{a}\ \rhd\ x \simeq \mathsf{a}}\ \textsc{Refl} \qquad \dfrac{}{x \mapsto \mathsf{a}\ \rhd\ x \simeq \mathsf{a}}\ \textsc{Refl}}{x \mapsto \mathsf{a}\ \rhd\ \mathsf{p}(x, x) \simeq \mathsf{p}(\mathsf{a}, \mathsf{a})}\ \textsc{Cong}}{\rhd\ (\text{let } x \simeq \mathsf{a} \text{ in } \mathsf{p}(x, x)) \simeq \mathsf{p}(\mathsf{a}, \mathsf{a})}\ \textsc{Let}
$$

matryoshka

SMT for HOL

Example: beta-reduction (1/2)

$$
\dfrac{
  \dfrac{}{\rhd\ \mathsf{a} \simeq \mathsf{a}}\ \textsc{Cong}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{x \mapsto \mathsf{a}\ \rhd\ \mathsf{p} \simeq \mathsf{p}}\ \textsc{Refl}
      \quad
      \dfrac{}{x \mapsto \mathsf{a}\ \rhd\ x \simeq \mathsf{a}}\ \textsc{Refl}
    }{x \mapsto \mathsf{a}\ \rhd\ \mathsf{p}\,x \simeq \mathsf{p}\,\mathsf{a}}\ \textsc{Cong}
    \quad
    \dfrac{}{x \mapsto \mathsf{a}\ \rhd\ x \simeq \mathsf{a}}\ \textsc{Refl}
  }{x \mapsto \mathsf{a}\ \rhd\ \mathsf{p}\,x\,x \simeq \mathsf{p}\,\mathsf{a}\,\mathsf{a}}\ \textsc{Cong}
}{\rhd\ (\lambda x.\, \mathsf{p}\,x\,x)\,\mathsf{a}\ \simeq \mathsf{p}\,\mathsf{a}\,\mathsf{a}}\ \textsc{Beta}
$$

Example: beta-reduction (2/2)

$$
\dfrac{
  \dfrac{
    \dfrac{}{\Gamma_1 \;\rhd \mathsf{f} \simeq \mathsf{f}}\,\textsc{Refl}
    \quad
    \dfrac{}{\Gamma_1 \;\rhd x \simeq w}\,\textsc{Refl}
  }{\Gamma_1 \;\rhd \mathsf{f}\, x \simeq \mathsf{f}\, w}\,\textsc{Cong}
  \qquad
  \dfrac{
    \dfrac{}{\Gamma_2 \;\rhd y \simeq \mathsf{f}\, w}\,\textsc{Refl}
    \quad
    \dfrac{}{\Gamma_3 \;\rhd \mathsf{p}\, z \simeq \mathsf{p}\,(\mathsf{f}\, w)}\,\textsc{Refl}
  }{\Gamma_2 \;\rhd (\lambda z.\, \mathsf{p}\, z)\, y \simeq \mathsf{p}\,(\mathsf{f}\, w)}\,\textsc{Beta}
}{
  \dfrac{\Gamma_1 \;\rhd (\lambda y.\, (\lambda z.\, \mathsf{p}\, z)\, y)\,(\mathsf{f}\, x) \simeq \mathsf{p}\,(\mathsf{f}\, w)}{\rhd (\lambda x.\, (\lambda y.\, (\lambda z.\, \mathsf{p}\, z)\, y)\,(\mathsf{f}\, x)) \simeq (\lambda w.\, \mathsf{p}\,(\mathsf{f}\, w))}\,\textsc{Bind}
}\,\textsc{Beta}
$$

where

$$\Gamma_1 = w,\, x \mapsto w;\;\; \Gamma_2 = \Gamma_1,\, y \mapsto \mathsf{f}\, w;\text{ and }\; \Gamma_3 = \Gamma_2,\, z \mapsto \mathsf{f}\, w:$$

Proof-producing contextual recursion

function *process*(Δ, t)
  match t
    case x:
      return *build_var*(Δ, x)
    case $\mathsf{f}(\bar{t}_n)$:
      $\bar{\Delta}'_n \leftarrow (\mathit{ctx\_app}(\Delta, \mathsf{f}, \bar{t}_n, i))_{i=1}^n$
      return *build_app*$\big(\Delta, \bar{\Delta}'_n, \mathsf{f}, \bar{t}_n, (\mathit{process}(\Delta'_i, t_i))_{i=1}^n\big)$
    case $Qx.\, \varphi$:
      $\Delta' \leftarrow \mathit{ctx\_quant}(\Delta, Q, x, \varphi)$
      return *build_quant*(Δ, Δ', Q, x, φ, *process*(Δ', φ))
    case $\text{let } \bar{x}_n \simeq \bar{r}_n \text{ in } t'$:
      $\Delta' \leftarrow \mathit{ctx\_let}(\Delta, \bar{x}_n, \bar{r}_n, t')$
      return *build_let*(Δ, Δ', $\bar{x}_n$, $\bar{r}_n$, t', *process*(Δ', t'))

▷ Parameterized by a notion of context and plugin functions

Example of plugin functions for theory simplification

For simplifying $u + 0$ and $0 + u$ to u:

Context Γ is a list of variables

function *ctx_quant*(Γ, Q, x, φ)
  return Γ, x

function *build_quant*$(\Gamma, \Gamma', Q, x, \varphi, \psi)$
  apply$(\textsc{Bind}, 1, \Gamma, Qx.\,\varphi, Qx.\,\psi)$
  return $Qx.\,\psi$

function *build_app*$(\Gamma, \bar{\Gamma}'_n, \mathsf{f}, \bar{t}_n, \bar{u}_n)$
  apply$(\textsc{Cong}, n, \Gamma, \mathsf{f}(\bar{t}_n), \mathsf{f}(\bar{u}_n))$
  if $\mathsf{f}(\bar{u}_n)$ *has form* $u + 0$ *or* $0 + u$ **then**
    apply$(\textsc{Taut}_+, 0, \Gamma, \mathsf{f}(\bar{u}_n), u)$
    apply$(\textsc{Trans}, 2, \Gamma, \mathsf{f}(\bar{t}_n), u)$
    return u
  else
    return $\mathsf{f}(\bar{u}_n)$

Theoretical properties

- ▷ Soundness of inference rules proven through an encoding into simply typed λ-calculus

$$M \ ::= \ \boxed{t} \mid \lambda x.\, M \mid (\lambda \bar{x}_n.\, M)\, \bar{t}_n$$

$$\frac{\mathcal{D}_1 \qquad \cdots \qquad \mathcal{D}_n}{M \simeq N}\,\mathsf{R}$$

- ▷ Correctness of proof-producing contextual recursion algorithm
- ▷ Cost of proof production is linear and of proof checking is (almost) linear*

  * assuming suitable data structures

Implementation

Proof output for veriT

Framework implemented with a proof-producing contextual recursion algorithm

- ⊕ fine-grained proofs for most processing transformations
- ⊕ No negative impact on performance
- ⊕ More transformations in proof producing mode
- ⊕ Dramatic simplification of the code base

Prototype checker in Isabelle/HOL

Maps proofs into Isabelle theorems

- ⊕ Judgements encoded in λ-calculus

Conclusions

- ▷ Centralizes manipulation of bound variables and substitutions
- ▷ Accommodates many transformations (e.g. Skolemization)
- ▷ Proof checking is (almost) linear
- ▷ Implementation and integration within veriT

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Future work

- ▷ Support global rewritings within the framework
- ▷ Implement mechanized proof reconstruction

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