Subtropical Satisfiability

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Presentation only (accepted FroCoS 2017)

SMT + non linear arithmetics

- High demand for non linear arithmetic reasoning capability
- Theory of real closed fields:
  decidable (QE: CAD, virtual substitution,. . .)
- Doubly exponential (existential fragment also high complexity)
- Complete decision procedure not always efficient enough
- Need for good heuristics

Our contribution

Simple heuristic to quickly discharge many proof obligations
(or failing quickly)

- Based on subtropical method: quickly find positive solution for $f = 0$ where f has hundreds of thousand of monomials, with dozen variables, degrees around 10 in each variable
- Here: find real solution for $f_1 > 0 \wedge \cdots \wedge f_n > 0$

Subtropical method: univariate case

Consider $x \geq 0$,

- $1 - 2x + x^3 > 0$

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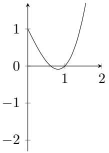

$y = 1 - 2x + x^3$

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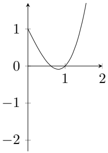

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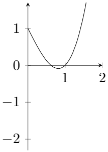

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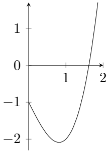

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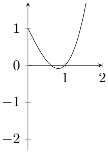

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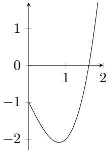

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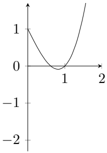

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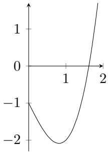

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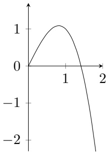

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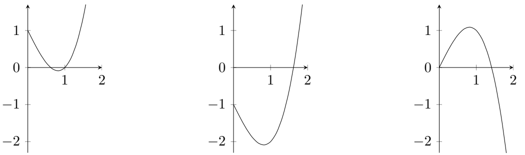
Find a model for $f > 0$?



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Find a model for $f > 0$?
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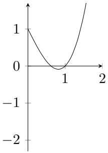

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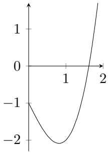

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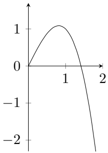

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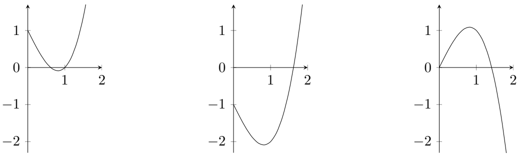

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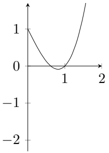

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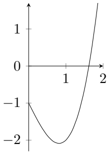

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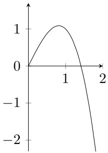

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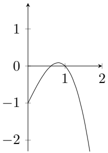

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Find a model for $f > 0$?
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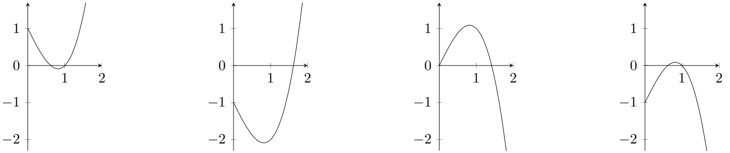
Incomplete: $-1 + 2x - x^3 > 0$? ($x = 0.8$)
But certainly quick



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Subtropical method: towards the multivariate case

Polynomial $-2x_1^5 + x_1^2 x_2 - 3x_1^2 - x_2^3 + 2x_2^2$ can be

- negative, e.g. $-2x_1^5$ dominates if x_1 large enough w.r.t. x_2
- positive, e.g. $2x_2^2$ dominates if x_2 small enough (not zero) x_2 and an even smaller x_1.

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- reduce to univariate, setting all variables but one to 0
- consider monomial of highest/lowest total degree (if unique)
- ordering? lexicographic?

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Contribution

monotonic total preorders on the exponent vectors

Strictly max. monomials (w.r.t. these preorders) can dominate, for suitable (positive) values of variables.

Subtropical method: towards the multivariate case (2)

A reminder of the original method

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Equivalent to consider the vertices of the Newton polytope, i.e. the set of exponent vectors.

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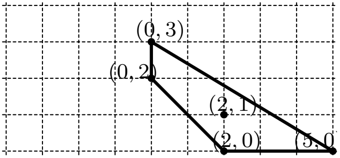

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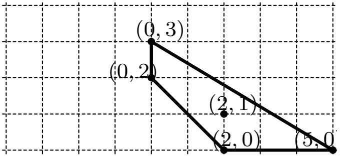

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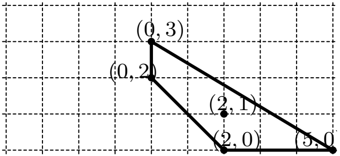

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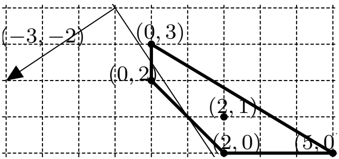

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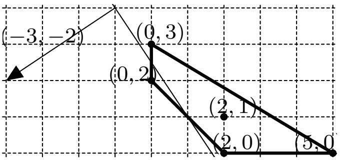
- Normal vector of separating plane provides witnesses
- E.g. $f > 0$ for $x_1 = t^{-3}$, $x_2 = t^{-2}$ with t sufficiently large

$$f = -2x_1^5 + x_1^2x_2 - 3x_1^2 - x_2^3 + 2x_2^2$$



Subtropical method: from preorders to QF_LRA SMT

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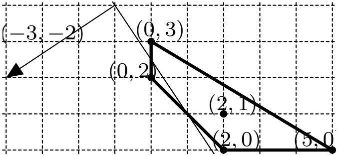


monotonic total preorders correspond to normal vectors

- $(x_1, x_2) \preceq (x_1', x_2')$ iff $-3x_1 - 2x_2 \leq -3x_1' - 2x_2'$
- $(5,0) \prec (2,1) \prec (2,0) \approx (0,3) \prec (0,2)$

To QF_LRA SMT?

- $\mathcal{S}^+ = \{(2,1),(0,2)\}$, $\mathcal{S}^- = \{(5,0),(2,0),(0,3)\}$

$$f > 0 \longleftrightarrow \bigwedge_{(p_1,p_2)\in\mathcal{S}^-} p_1 n_1 + p_2 n_2 + c < 0 \wedge \bigvee_{(p_1,p_2)\in\mathcal{S}^+} p_1 n_1 + p_2 n_2 + c > 0$$

- linear constraints on real variables, n_1, n_2, c

Several polynomials

One polynomial:

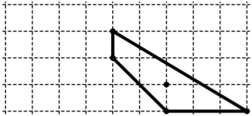
$$f_1 = -2x_1^5 + x_1^2 x_2$$
$$-3x_1^2 - x_2^3 + 2x_2^2$$

Several polynomials

One polynomial:

- build the Newton polytope

$$f_1 = -2x_1^5 + x_1^2x_2 \\ -3x_1^2 - x_2^3 + 2x_2^2$$



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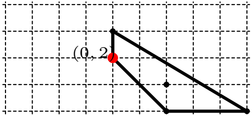

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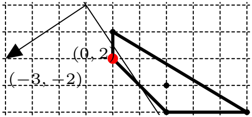

$f_1 > 0$
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(t sufficiently large)

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Several polynomials:

$$f_1 = -2x_1^5 + x_1^2 x_2 \\ -3x_1^2 - x_2^3 + 2x_2^2$$

$$f_2 = 1 - x_1 x_2 \\ -5x_1 - 6x_2$$

$$f_3 = x_1 x_2 \\ -x_1^5 x_2^2 + x_1 x_2^4$$

Several polynomials

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Several polynomials:

- build the Newton polytope**s**

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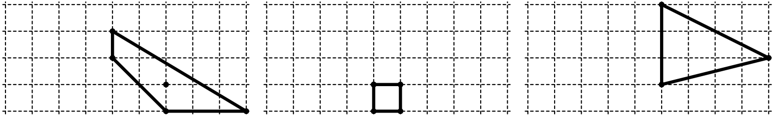

Several polynomials

One polynomial:

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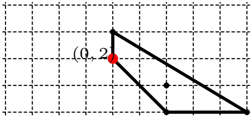
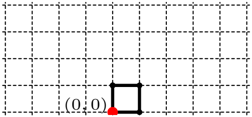
Several polynomials:

- build the Newton polytope**s**
- find suitable vertice**s**

$$f_1 = -2x_1^5 + x_1^2x_2 - 3x_1^2 - x_2^3 + 2x_2^2$$



$$f_2 = 1 - x_1x_2 - 5x_1 - 6x_2$$



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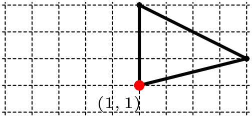

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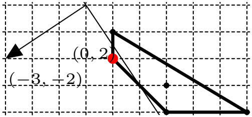

$f_1 > 0$
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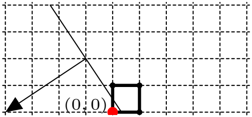
(t sufficiently large)

$$f_2 = 1 - x_1 x_2 - 5x_1 - 6x_2$$



$f_2 > 0$
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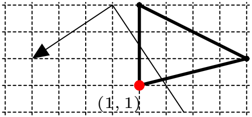
(t sufficiently large)

$$f_3 = x_1 x_2 - x_1^5 x_2^2 + x_1 x_2^4$$



$f_3 > 0$
if $x_1 = t^{-3}$, $x_2 = t^{-2}$
(t sufficiently large)

Several polynomials

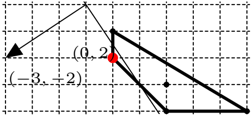
One polynomial:

- build the Newton polytope
- find a suitable vertex
- normal vector to separating plane provides witness

Several polynomials:

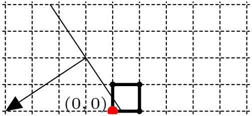
- build the Newton polytope**s**
- find suitable vertice**s**
- normal vector to separating plane provides witness

$$f_1 = -2x_1^5 + x_1^2 x_2 - 3x_1^2 - x_2^3 + 2x_2^2$$



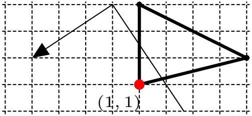
$f_1 > 0$
if $x_1 = t^{-3}$, $x_2 = t^{-2}$
(t sufficiently large)

$$f_2 = 1 - x_1 x_2 - 5x_1 - 6x_2$$



$f_2 > 0$
if $x_1 = t^{-3}$, $x_2 = t^{-2}$
(t sufficiently large)

$$f_3 = x_1 x_2 - x_1^5 x_2^2 + x_1 x_2^4$$



$f_3 > 0$
if $x_1 = t^{-3}$, $x_2 = t^{-2}$
(t sufficiently large)

common normal vector ensures existence of global solution

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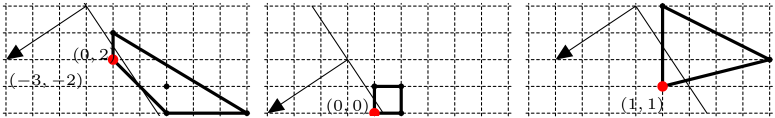
Several polynomials:

- build the Newton polytope**s**
- find suitable vertice**s**
- normal vector to separating plane provides witness

$$f_1 = -2x_1^5 + x_1^2 x_2 - 3x_1^2 - x_2^3 + 2x_2^2 \qquad f_2 = 1 - x_1 x_2 - 5x_1 - 6x_2 \qquad f_3 = x_1 x_2 - x_1^5 x_2^2 + x_1 x_2^4$$



$f_1 > 0$ if $x_1 = t^{-3}$, $x_2 = t^{-2}$ (t sufficiently large)

$f_2 > 0$ if $x_1 = t^{-3}$, $x_2 = t^{-2}$ (t sufficiently large)

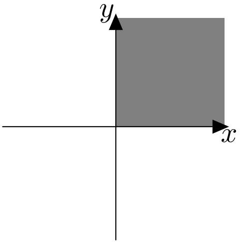
$f_3 > 0$ if $x_1 = t^{-3}$, $x_2 = t^{-2}$ (t sufficiently large)

common normal vector ensures existence of global solution

n polynomial constraints? Conjunction of n QF_LRA problems sharing only variables to describe normal vector

From positive to arbitrary solution

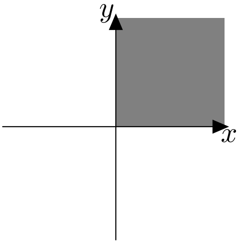
- Up to now: $\bigwedge_i f_i > 0$ with all $\bigwedge_i x_i > 0$



$f(x, y) > 0$

From positive to arbitrary solution

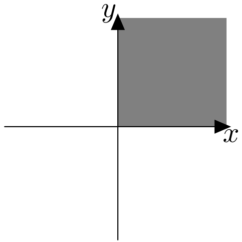
- Up to now: $\bigwedge_i f_i > 0$ with all $\bigwedge_i x_i > 0$
- Removing the condition $\wedge_i \; x_i > 0$?



$f(x,y) > 0$

From positive to arbitrary solution

- Up to now: $\bigwedge_i f_i > 0$ with all $\bigwedge_i x_i > 0$
- Removing the condition $\wedge_i\ x_i > 0$?
- Just consider every hyper-quadrant



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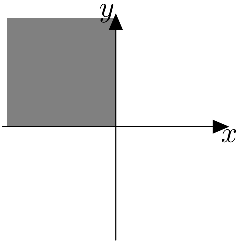

$f(x, y) > 0$

From positive to arbitrary solution

- Up to now: $\bigwedge_i f_i > 0$ with all $\bigwedge_i x_i > 0$
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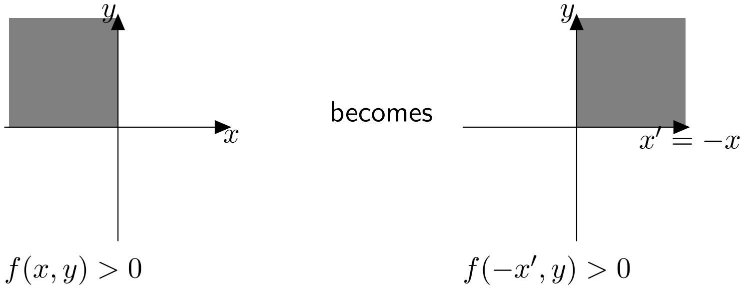
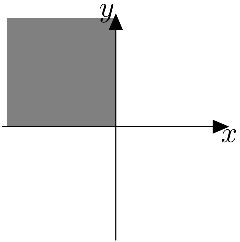


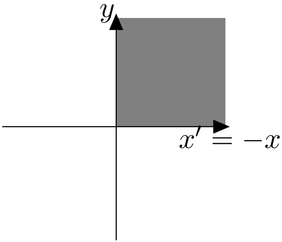
From positive to arbitrary solution

- Up to now: $\bigwedge_i f_i > 0$ with all $\bigwedge_i x_i > 0$
- Removing the condition $\wedge_i\ x_i > 0$?
- Just consider every hyper-quadrant
- This can be encoded into the QF_LRA SMT problem; no need to check 2^n formulas



$f(x, y) > 0$

becomes



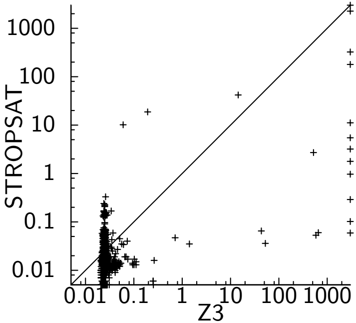
$f(-x', y) > 0$

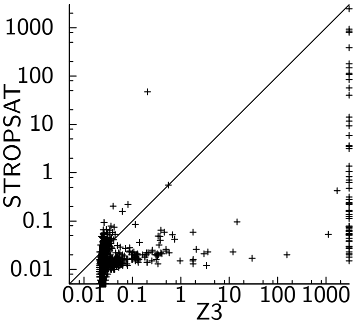
Experimental results

- STROPSAT integrated in veriT (not the SMT-COMP version)
- Tested on SMT-LIB/QF_NRA on suitable problems,
  i.e. 4917/11601 files: 3265 sat, 106 unknown, 1546 unsat
- CVC4 used to handle linear solving
- 2500s timeout, 20GB

On 1546 unsat-labeled formulas: 200 unsat by LRA, cumulative time to fail on the 1346 others: 18.45s, max 0.1s
Shows satisfiability for 2403 problems, including 15 "unknown" problems (and 9 where Z3 fails)

Experimental results





When STROPSAT does not fail

- time comparable to Z3
- sometimes succeeds alone
- if timeouts, Z3 too

STROPSAT is quick to fail

Conclusion

- A heuristic, providing quick solutions, or failing quickly
- Good results for many SMT benchmarks
- Not sensitive to the number of variables; actually, gets "better" when the number of variables grows
- Investigate its use in context where getting models is paramount, i.e. testing phase of raSAT loop
- What can we do along these lines to help complete decision procedures?
- Better understand when the method works

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